CORRIGE EXERCICE No 10

A.

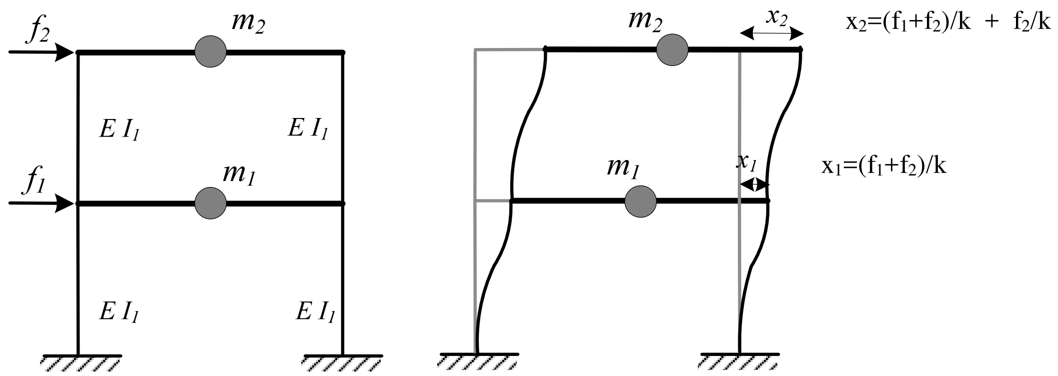


Figure. Déplacements des étages.

1.
Avec une répartition linéaire de forces sur les deux étages, on a :

$$\begin{cases} f_1 = F \\ f_2 = 2F \end{cases}$$

$$\begin{cases} x_1 = \dfrac{3F}{k} \\ x_2 = \dfrac{5F}{k} \end{cases}$$

Le quotient de Rayleigh s'écrit comme suit :

$$\omega_R^2 = \frac{\sum_j F_j x_j}{\sum_j m_j x_j^2}$$

Ce qui donne :

$$\sum_j F_j x_j = F \cdot \frac{3F}{k} + 2F \cdot \frac{5F}{k} = \frac{F^2}{k}(3+10)$$

$$\sum_j m_j x_j^2 = m \cdot \frac{9F^2}{k^2} + m \cdot \frac{25F^2}{k^2} = m\frac{F^2}{k^2}(9+25)$$

Avec :

$$m_1 = m_2 = 20000\, kg \text{ et } k_1 = k_2 = k = 2\frac{12EI_1}{h^3} = 180.10^5\ N/m$$

$$\omega_R^2 = \frac{13}{34} \cdot \frac{k}{m} = 344.2$$

$$\omega_R = 18.55\ \ rad/s$$

$$f_n = 2.952 Hz$$

2.

$$\begin{cases} f_1 = mg \\ f_2 = mg \end{cases}$$

$$\begin{cases} x_1 = \dfrac{2mg}{k} = 0.022m \\ x_2 = \dfrac{3mg}{k} = 0.033m \end{cases}$$

L'évaluation du quotient de Rayleigh par la formule simplifiée donne :

$$T_1[s] = 2 \cdot \sqrt{x_N[m]} = 2 \cdot \sqrt{0.033} = 2 \cdot 0.182 = 0.364\,s$$

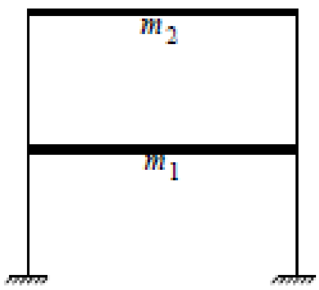
$$\omega_1 = \frac{2 \cdot \pi}{T_1} = 17.26\ rad/s$$

$$f_n = 2.747\,Hz$$

3. L'estimation de la fréquence fondamentale de la structure obtenue en appliquant des forces réparties linéairement est très proche de la fréquence exacte $f_n = 2.951Hz$ ($\omega_1 = 18.54\ rad/s$). La formule simplifiée du quotient de Rayleigh est moin précise.

B.

Calcul des fréquences propres d'un système à deux degrés de liberté
Une adaptation de la méthode des déplacements est utilisée.



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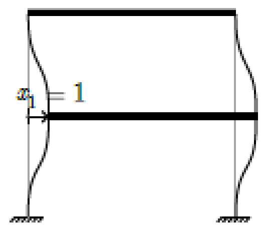
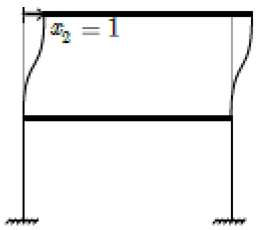
<tr><td>Effets</td><td>
</td><td>
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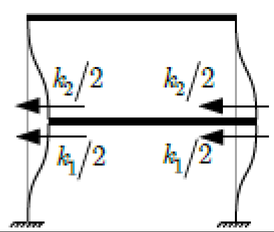
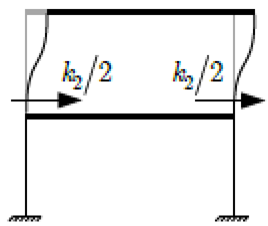
<tr><td>Selon x_1</td><td>
</td><td>
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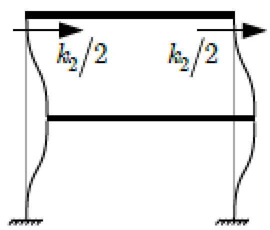
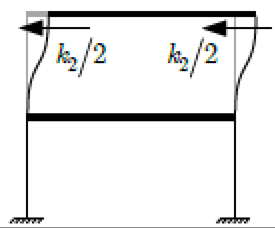
<tr><td>Selon x_2</td><td>
</td><td>
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Sous la forme matricielle le système d'équations s'écrit :

$$\left\{\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} - \omega_n^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\right\}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Afin de trouver les valeurs propres le déterminant doit s'annuler :

$$\det\begin{vmatrix} k_1+k_2-\omega_n^2 m_1 & -k_2 \\ -k_2 & k_2-\omega_n^2 m_2 \end{vmatrix} = 0 \quad \Rightarrow m_1 m_2 \omega_n^4 - \left[(k_1+k_2)m_2 + k_2 m_1\right]\omega_n^2 + k_1 k_2 = 0$$

Avec $m_1 = 2m_2 = 2m$ et $k_1 = 1.5k_2 = 1.5k \,(m = 2000t \quad et \quad k = 40000kN/m)$

$$\omega_n^4 - \frac{9k}{4m}\omega_n^2 + \frac{3}{4}\cdot\left(\frac{k}{m}\right)^2 = 0$$

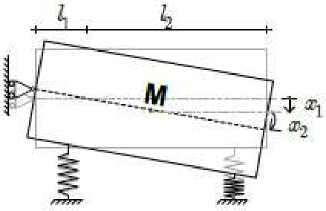
Les solutions sont :

$$\omega_n = \sqrt{\left(\frac{9\pm\sqrt{33}}{8}\right)\cdot\left(\frac{k}{m}\right)} = \sqrt{\left(\frac{9\pm\sqrt{33}}{8}\right)\cdot\left(\frac{40000}{2000}\right)} \Rightarrow \omega_{n,1} = 2.85\frac{rad}{s};\ \omega_{n,2} = 6.07\frac{rad}{s}$$

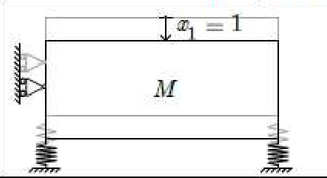
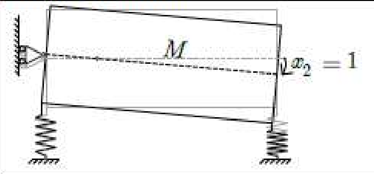
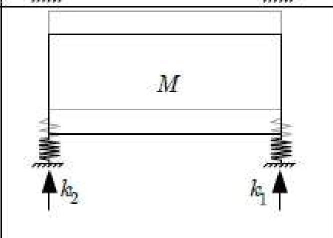
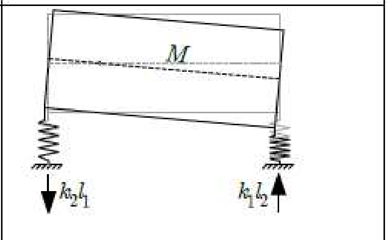
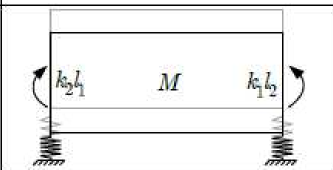
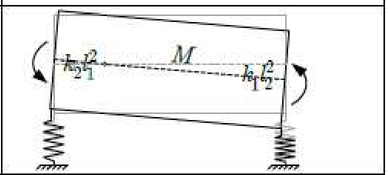
C.

1. (a) 2 degrés de liberté.

   (b)



   (c)

	Mouvement unitaire dans le sens et à la position qui correspond à 1 degré de liberté	
Effets		
Selon x_1		
Selon x_2		

$$\underline{K}_{mn} = \begin{bmatrix} k_1 + k_2 & k_1 l_2 - k_2 l_1 \\ k_1 l_2 - k_2 l_1 & k_2 l_1^2 + k_1 l_2^2 \end{bmatrix}$$

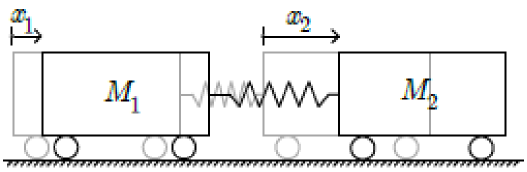
$$\underline{M}\ddot{\underline{X}} + \underline{K}_{mn}\underline{X} = 0$$

Soit $$\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} K_1 + K_2 & K_1 l_2 - K_2 l_1 \\ K_1 l_2 - K_2 l_1 & K_1 l_2^2 + K_2 l_1^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

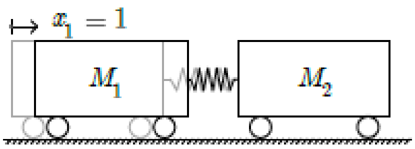
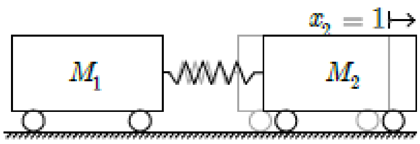
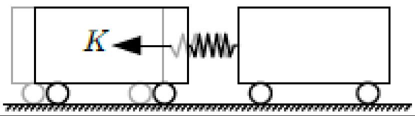
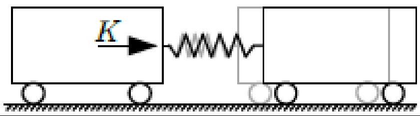
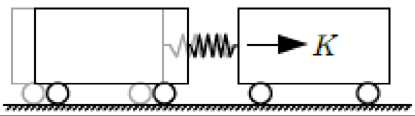
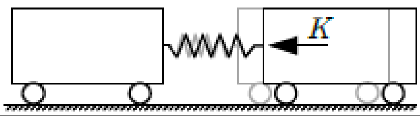
Remarque :
Il faut noter qu'ici, contrairement à la méthode utilisée dans le cours de Mécanique des structures IV, il n'y a pas d'inversion de la matrice des rigidités. (Voir Annexe A du polycopié)

2. (a) 2 degrés de liberté.

(b)



(c)

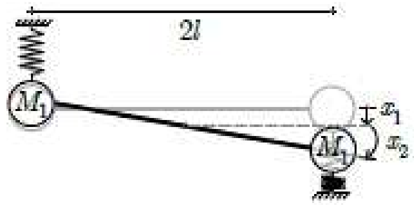
	Mouvement unitaire dans le sens et à la position qui correspond à 1 degré de liberté	
Effets		
Selon x_1		
Selon x_2		

$$\underline{K}_{mn} = \begin{bmatrix} K & -K \\ -K & K \end{bmatrix}$$

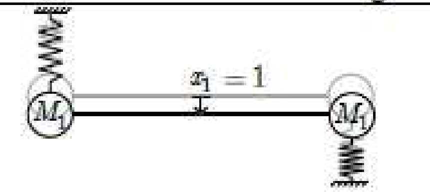
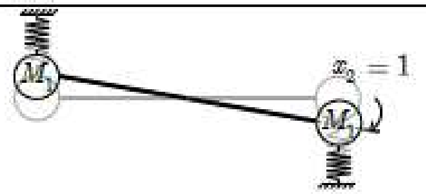
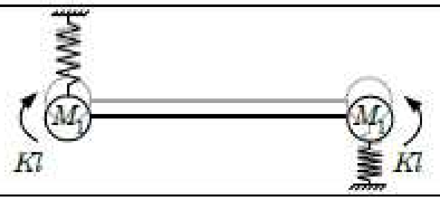
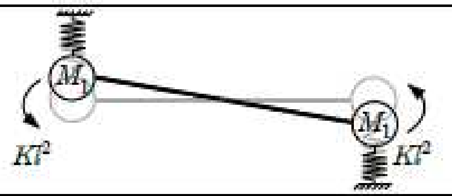
$$\underline{M}\,\ddot{\underline{X}} + \underline{K}_{mn}\,\underline{X} = 0$$

Soit
$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} K & -K \\ -K & K \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

3. (a) 2 degrés de liberté.
(b)



(c)

	Mouvement unitaire dans le sens et à la position qui correspond à 1 degré de liberté	
Effets		
Selon x_1		
Selon x_2		

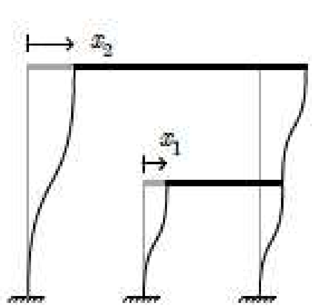
$$\underline{K}_{mn} = \begin{bmatrix} 2K & 0 \\ 0 & 2Kl^2 \end{bmatrix}$$

$$\underline{M}\ddot{\underline{X}} + \underline{K}_{mn}\underline{X} = 0$$

Soit $$\begin{bmatrix} 2M_1 & 0 \\ 0 & 2M_1 l^2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2K & 0 \\ 0 & 2Kl^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

4. (a) 2 degrés de liberté.
(b)



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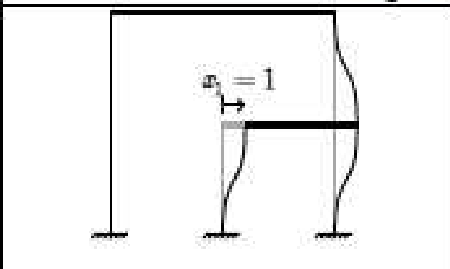
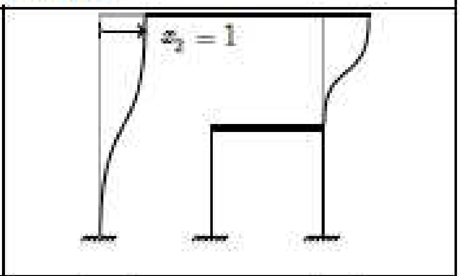
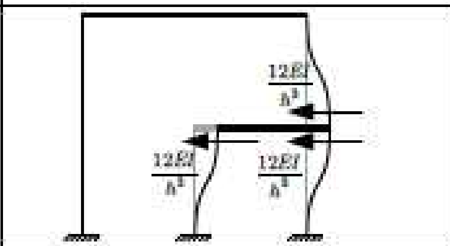
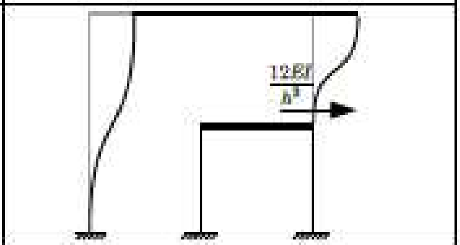
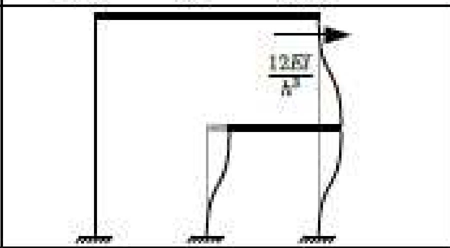
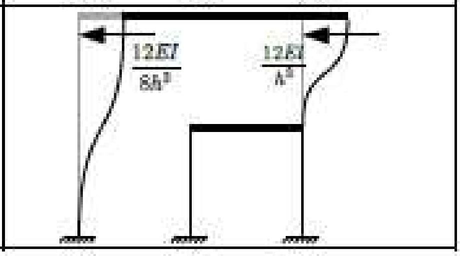
	Mouvement unitaire dans le sens et à la position qui correspond à 1 degré de liberté	
Effets		
Selon x_1		
Selon x_2		

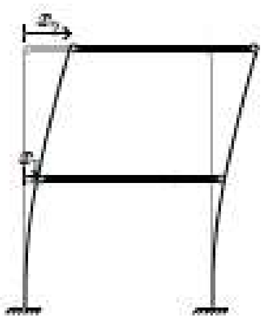
$$si \qquad k = \frac{12EI}{h^3}$$

$$\underline{K}_{mn} = \begin{bmatrix} 3k & -k \\ -k & \frac{9}{8}k \end{bmatrix}$$

$$\underline{M}\underline{\ddot{X}} + \underline{K}_{mn}\underline{X} = 0$$

$$Soit \quad \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 24k & -8k \\ -8k & 9k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

5. (a) 2 degrés de liberté.
(b)



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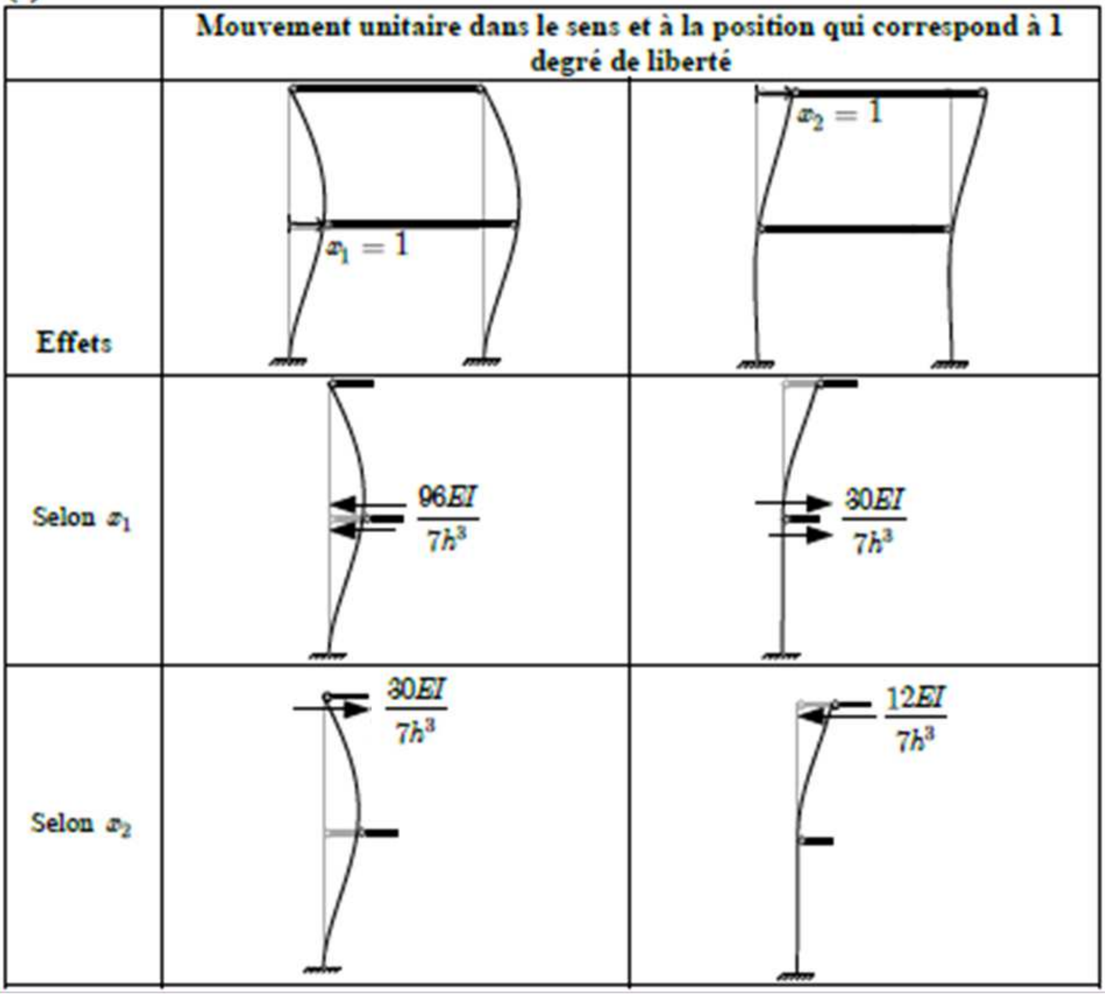

$$\underline{K}_{mn} = \frac{6EI}{7h^3}\begin{bmatrix} 16 & -5 \\ -5 & 2 \end{bmatrix}$$

$$\underline{M\ddot{X}} + \underline{K}_{mn}\,\underline{X} = 0$$

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \frac{6EI}{7h^3}\begin{bmatrix} 16 & -5 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$